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κ-Lindelöf κ-stratifiable spaces are κ-metrizable

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Abstract

A (locally) κ-Lindelöf κ-stratifiable space is shown to be κ-metrizable. Since the (local) κ-Lindelöf property is a weaker condition than compactness, the result generalizes Ceder's and Vaughan's theorems that compact stratifiable and compact linearly stratifiable spaces are metrizable.

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A stratifiable space is a T_2 space with a function assigning to each closed set H a sequence of open sets $\{G_n(H) \mid n \in \omega\}$ such that (1) $H = \bigcap_{n\in\omega} G_n(H) = \bigcap_{n\in\omega} \overline{G_n(H)}$ and (2) if $H \subseteq K$ with H and K closed, then $G_n(H) \subseteq G_n(K)$ for each n. Ceder [2] proved that a compact (actually, locally compact) stratifiable space is metrizable.

For an infinite cardinal κ, a T_2 space is stratifiable over κ [5] if there is a function which assigns to each closed set H a κ-sequence of open sets $\{G_\alpha(H) \mid \alpha < \kappa\}$ for which:

(1κ) $H = \bigcap_{\alpha<\kappa} G_\alpha(H) = \bigcap_{\alpha<\kappa} \overline{G_\alpha(H)}$,

(2κ) if $H \subseteq K$ with H and K closed, then $G_\alpha(H) \subseteq G_\alpha(K)$ for each $\alpha < \kappa$, and

(3κ) if $\alpha < \delta < \kappa$, then $G_\delta(H) \subseteq G_\alpha(H)$.

A space is κ-stratifiable if it is stratifiable over κ but is not stratifiable over any infinite cardinal less than κ.

For a κ-stratifiable space Vaughan [5] proved κ must be a regular cardinal and any subset of the space with fewer than κ elements is closed and discrete. Consequently, any property which requires countable sets to have limit points will force a κ-stratifiable space to be stratifiable. For example, a compact κ-stratifiable space is metrizable. Using a property weaker than compactness we can generalize Vaughan's result.

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A space is κ-Lindelöf (also known as κ-compact or finally κ-compact) if every open cover has a subcover with cardinality less than κ. In a κ-Lindelöf space, any set with at least κ elements must have a limit point. In a locally κ-Lindelöf space every point has an open neighborhood N such that any subset of N with at least κ elements has a limit point in N. We will see that this property is sufficient to cause a κ-stratifiable space to be κ-metrizable. Recall that, for a regular infinite cardinal κ, a space is κ-metrizable if it has a decreasing uniform base of the form $\{U_\alpha \mid \alpha < \kappa\}$. Many people have worked with κ-metrizable spaces proving generalizations of many of the standard metrization theorems. We will use Reichel's [4] generalization of Bing's theorem that a paracompact Moore space is metrizable and the following higher cardinal version of conditions given by Heath and Hodel [3].

Lemma. *Suppose X is stratifiable over the regular cardinal κ.*

(1) *If x belongs to $G_\alpha(\{x_\alpha\})$ for each $\alpha < \kappa$, then (x_α) converges to x.*

(2) *If y_α belongs to $G_\alpha(\{x_\alpha\})$ for each $\alpha < \kappa$ and y is a (complete) accumulation point of $\{y_\alpha \mid \alpha < \kappa\}$, then y is a (complete) accumulation point of $\{x_\alpha \mid \alpha < \kappa\}$.*

Proof. Let X be stratifiable over regular cardinal κ and, for each closed set H, let $\{G_\alpha(H) \mid \alpha < \kappa\}$ be the κ-sequence of open sets assigned by the κ-stratification function.

To prove (1), let x belong to $G_\alpha(\{x_\alpha\})$ for each $\alpha < \kappa$, and suppose there were an open neighborhood U of x such that, for each α, some x_{γ_α} with γ_α greater than α does not belong to U. Then, since x is in $G_\delta(\overline{\{x_{\gamma_\alpha} \mid \alpha < \kappa\}})$ for each δ, the point x belongs to $\overline{\{x_{\gamma_\alpha} \mid \alpha < \kappa\}}$ which is contained in $X \backslash U$. That is a contradiction. To prove (2), suppose y_α belongs to $G_\alpha(\{x_\alpha\})$ for each $\alpha < \kappa$ and y is a (complete) accumulation point of $\{y_\alpha \mid \alpha < \kappa\}$ but y is not a (complete) accumulation point of $\{x_\alpha \mid \alpha < \kappa\}$. Some open neighborhood U of y contains no x_α different from y. For each α, the point y_α belongs to the subset $G_\alpha(\{x_\alpha\})$ of $G_\alpha(X \backslash U)$. Since the κ-stratification elements decrease, we have $\{y_\alpha \mid \alpha > \delta\}$ contained in $G_\delta(X \backslash U)$ for each $\delta < \kappa$. Then y belongs to $X \backslash U$. That is a contradiction.

Theorem 2 (Reichel). *For $\kappa > \omega$, X is κ-metrizable if and only if*

(1) *X is a κ-additive space (i.e., the intersection of fewer than κ open sets is open),*

(2) *X is collectionwise normal, and*

(3) *there is a κ-sequence $\{\mathcal{G}_\alpha \mid \alpha < \kappa\}$ of open covers such that at each point x of X the collection $\{\mathrm{St}(x, \mathcal{G}_\alpha) \mid \alpha < \kappa\}$ is a local base.*

Theorem. *A locally κ-Lindelöf κ-stratifiable space is κ-metrizable.*

Proof. Let X be a locally κ-Lindelöf κ-stratifiable space. Since a κ-stratifiable space is paracompact [5], then X is collectionwise normal.

Although the κ-stratification elements usually do not form a local base at x, they do in a κ-Lindelöf κ-stratifiable space. We will now show by contradiction that, in a locally κ-Lindelöf, κ-stratifiable space, the traces of the κ-stratification elements $\{G_\alpha(\{x\}) \mid \alpha <$

$\kappa\}$ on a neighborhood $L(x)$ witnessing the local κ-Lindelöf property form a local base at x. For proof, let $L(x)$ be an open neighborhood of x such that any subset of $L(x)$ of cardinality at least κ has a limit point in $L(x)$. Suppose there is an open neighborhood O of x such that, for each $\alpha < \kappa$, some element y_α belongs to $G_\alpha(\{x\}) \cap L(x)$ but not to O.

If $|\{y_\alpha \mid \alpha < \kappa\}| < \kappa$, then some $y_{\alpha'}$ occurs κ many times and, by property (3κ) above, $y_{\alpha'}$ belongs to $\bigcap_{\alpha<\kappa} G_\alpha(\{x\})$. This is a contradiction since $y_{\alpha'}$ in $X\backslash O$ cannot be equal to x in O. Then $\{y_\alpha \mid \alpha < \kappa\}$ has cardinality κ. Let $\{y_{\alpha_\delta} \mid \delta < \kappa\}$ be a κ-subsequence of distinct points of $\{y_\alpha \mid \alpha < \kappa\}$. Since all y_α's are in $L(x)$, the set $\{y_{\alpha_\delta} \mid \delta < \kappa\}$ has a limit point p which belongs to $L(x)$ but not to O. For each δ we have y_{α_δ} in $G_{\alpha_\delta}(\{x\})$ so that $\overline{\{y_{\alpha_\delta} \mid \delta \geqslant \theta\}}$ is contained in $\overline{G_{\alpha_\theta}(\{x\})}$ for any $\theta < \kappa$. Since $\{y_{\alpha_\delta} \mid \delta \leqslant \theta\}$ is closed and discrete, p must belong to $\overline{\{y_{\alpha_\delta} \mid \delta \geqslant \theta\}}$ and therefore to $\overline{G_{\alpha_\theta}(\{x\})}$ for any θ. By (3κ) p belongs to $\bigcap_{\alpha<\kappa} \overline{G_\alpha(\{x\})}$ and thus must be x. That is a contradiction because x belongs to O while p does not. Thus, some $G_\alpha(\{x\}) \cap L(x)$ is contained in O.

To show that X is a κ-additive space: suppose $\{U_\rho \mid \rho < \theta\}$ is a collection of open sets with $\theta < \kappa$ and suppose x belongs to $\bigcap_{\rho<\theta} U_\rho$. For each ρ there is a α_ρ for which $G_{\alpha_\rho}(\{x\}) \cap L(x)$ contains x and is contained in U_ρ. For $\delta = \sup\{\alpha_\rho \mid \rho < \theta\}$ the set $G_\delta(\{x\}) \cap L(x)$ contains x and is contained in $\bigcap_{\rho<\theta} U_\rho$. Thus $\bigcap_{\rho<\theta} U_\rho$ is open.

Let $\mathcal{L} = \{L(x) \mid x \in X\}$ where for each x the neighborhood $L(x)$ of x witnesses the local κ-Lindelöf property at x as above. Since X is paracompact, there is a locally finite open refinement $\mathcal{M}$ of $\mathcal{L}$ which covers X. For each α, let $\mathcal{N}_\alpha$ be the family $\{G_\alpha(\{x\}) \cap (\bigcap \mathcal{M}(x)) \mid x \in X\}$ where $\mathcal{M}(x) = \{M \in \mathcal{M} \mid x \in M\}$. We will prove by contradiction that for each x the collection $\{\mathrm{St}(x, \mathcal{N}_\alpha) \mid \alpha < \kappa\}$ is a local base at x. Suppose O is an open neighborhood of x and for every α there is a y_α belonging to $\mathrm{St}(x, \mathcal{N}_\alpha)$ but not to O. For some x_α, the set $G_\alpha(\{x_\alpha\}) \cap (\bigcap \mathcal{M}(x_\alpha)$ contains both x and y_α. By part (1) of the lemma, the κ-sequence (x_α) converges to x. Suppose γ is less than κ. Then x belongs to $\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}}$ and, for each $\delta > \gamma$, the point y_δ belongs to $G_\delta(\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}})$. Since the κ-stratification elements decrease, the set $\{y_\mu \mid \mu \geqslant \delta\}$ is contained in $G_\delta(\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}})$. If $|\{y_\alpha \mid \alpha < \kappa\}| < \kappa$, then some y_{α_0} occurs κ many times. For each δ the point y_{α_0} belongs to $\{y_\mu \mid \mu \geqslant \delta\}$ and thus to $G_\delta(\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}})$. The point y_{α_0} belongs to $\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}}$ for each $\gamma < \kappa$ and therefore is x. That is a contradiction since x belongs to O while y_{α_0} does not. Then $|\{y_\alpha \mid \alpha < \kappa\}| = \kappa$. Without loss of generality, assume the y_α's are distinct. Since x belongs to each set in the point finite collection $\mathcal{M}$ to which any of the x_α's belong and since y_α also belongs to those sets in $\mathcal{M}$ that contain x_α, then some set M from $\mathcal{M}$ contains x and κ of the y_α's. Let $M \cap \{y_\alpha \mid \alpha < \kappa\} = \{y_{\alpha_\beta} \mid \beta < \kappa\}$. Since $\mathcal{M}$ refines $\mathcal{L}$, there is a set $L(w)$ in $\mathcal{L}$ which contains M. By our choice of $L(w)$, the set $\{y_{\alpha_\beta} \mid \beta < \kappa\}$ has a limit point p in $L(w)$. For each $\delta > \gamma$ the point p belongs to $\overline{\{y_{\alpha_\mu} \mid \mu \geqslant \delta\}}$ and thus to $\overline{G_\delta(\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}})}$. By ($1\kappa$) p belongs to $\overline{\{x_\alpha \mid \alpha \geqslant \gamma\}}$ for each γ and therefore is x. Again, that is a contradiction since x belongs to O while p belongs to $X\backslash O$. Thus, for some α, $\mathrm{St}(x, \mathcal{N}_\alpha)$ is contained in O. By Reichel's theorem X is κ-metrizable.

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